

Pfaffian Systems of A-Hypergeometric Equations

Takayuki Hibi, Kenta Nishiyama, Nobuki Takayama

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Abstract: We study bases of Pfaffian systems for A -hypergeometric system. Gröbner deformations give bases. For hypergeometric system associated to a class of order polytopes, these bases have a combinatorial description. These theoretical study gives us algorithms of transforming a given A -hypergeometric system into a Pfaffian system. The size of the Pfaffian system associated to a subclass of the order polytopes has the growth rate of the polynomial order.

1 Introduction

Let F be a vector valued function in x_1, \dots, x_n . We suppose the length of F is r . Let $P_i(x)$, $i = 1, \dots, n$ be $r \times r$ matrices satisfying

$$\frac{\partial P_i}{\partial x_j} + P_i P_j = \frac{\partial P_j}{\partial x_i} + P_j P_i$$

for all $i \neq j$. The system of linear differential equations

$$\frac{\partial F}{\partial x_i} = P_i(x)F, \quad i = 1, \dots, n$$

is called a *Pfaffian system*. We also call the system of the linear differential operators $\frac{\partial}{\partial x_i} - P_i$ a Pfaffian system. The number r is called the size of the Pfaffian system. For a given zero dimensional left ideal in the ring of differential operators with rational coefficients, it is well-known that an associated Pfaffian system can be obtained by a Gröbner basis method (see, e.g., [12, Appendix]). A geometric method to find a Pfaffian system associated to a given definite integral with parameters is the use of twisted cohomology groups (see, e.g., [2]).

In [12], new methods in statistics are proposed. They are holonomic gradient method (HGM) and holonomic gradient descent (HGD). The HGM is a method to evaluate numerically the normalization constant, which is a function of parameters x , for a given unnormalized probability distribution, and the HGD is a method to make a maximal likelihood estimate by utilizing the HGM. The key step of these methods is to construct a Pfaffian system associated to the normalization constant. The size of the Pfaffian system determines the complexity of the HGM and the HGD (see, e.g., [10]).

A -hypergeometric systems have been studied for the past 20 years (see, e.g., [5], [6], [15]) with several applications in various fields. We will add a new motivation to study it; we note that it can be regarded as a system of differential equations for normalization constants of a class of exponential family of probability distributions (see, e.g., (10)). Following these studies and the new motivation, we pose the following questions to apply A -hypergeometric systems to statistics.

1. Give an efficient method to find a Pfaffian system associated to a given A -hypergeometric system.
2. Find a nice subclass of A -hypergeometric systems of which associated Pfaffian systems have a moderate size.

We will give an answer to these questions in this paper. Let us be more precise. Let R_n be the ring of differential operators with rational function coefficients in n -variables. The first step to find a Pfaffian system associated to a given left ideal I in R_n is to give a basis of R_n/I as a $\mathbf{C}(x)$ -vector space. When the set $\{u_1, \dots, u_r\}$ is the basis, there exists a matrix $P_i(x)$ such that $\partial/\partial x_i U \equiv P_i U \pmod{I}$, $U = (u_1, \dots, u_r)^T$. We can show that $\partial/\partial x_i - P_i$ is a Pfaffian system. We call $\{u_1, \dots, u_r\}$ the basis of the Pfaffian system. We study bases of Pfaffian systems for A -hypergeometric systems. We will show that Gröbner deformations give bases (Theorem 1). For hypergeometric system associated to a class of order polytopes (see, e.g., [9]), these bases have a combinatorial description (Theorem 5). These theoretical study gives us algorithms of transforming a given A -hypergeometric system into a Pfaffian system (Section 6).

The size of the Pfaffian system associated to a subclass of the order polytopes has the growth rate of the polynomial order (Theorem 6). We expect that this together with our algorithm yields a new class of an exponential family of probability distributions for which we can apply the holonomic gradient method (HGM) and the holonomic gradient descent (HGD) efficiently.

2 Bases for the Pfaffian System

We denote by $A = (a_{ij})$ a $d \times n$ -matrix whose elements are integers. We suppose that the set of the column vectors of A spans \mathbf{Z}^d . Let s_1, \dots, s_d be indeterminates. Following notations in [15], we denote by $H_A[s]$ a left ideal generated in the Weyl algebra

$$D[s] = \mathbf{C}[s_1, \dots, s_d] \langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle, \quad \partial_i = \partial/\partial x_i$$

by

$$\sum_{j=1}^n a_{ij} x_j \partial_j - s_i, \quad (i = 1, \dots, d) \quad (1)$$

$$\prod_{i=1}^n \partial_i^{u_i} - \prod_{j=1}^n \partial_j^{v_j} \quad (2)$$

(with $u, v \in \mathbf{N}_0^n$ running over all u, v such that $Au = Av$).

Here, $\mathbf{N}_0 = \{0, 1, 2, \dots\}$. We call the ideal generated in $\mathbf{C}[\partial_1, \dots, \partial_n]$ by the elements of the form (2) the affine toric ideal and denote it by I_A . We denote by $E_i - s_i$ the operator (1). For complex parameters β_i , the system of linear differential equations $(E_i - \beta_i)f = 0$ ($i = 1, \dots, d$), $(\partial^u - \partial^v)f = 0$ ($Au = Av$) is called the A -hypergeometric system of differential equations or shortly the A -hypergeometric system. We also sometimes call the ideal $H_A[s]$ the A -hypergeometric system (with indefinite parameters).

Let R_n be the ring of differential operators with rational function coefficients

$$\mathbf{C}(s, x) \langle \partial_1, \dots, \partial_n \rangle. \quad (3)$$

We are interested in bases of $R_n/(R_n H_A[s])$ as the vector space over the field $\mathbf{C}(s, x)$. Any basis of the vector space yields an associated Pfaffian system or an integrable connection associated to $H_A[s]$. Let u_1, \dots, u_r be a basis of $R_n/(R_n H_A[s])$. For u_j , there exist rational functions $p_{ij}^k \in \mathbf{C}(s, x)$ such that $\partial_i u_j \equiv \sum_{k=1}^r p_{ij}^k u_k \pmod{R_n H_A[s]}$. The system of differential equations $\partial_i F = (p_{ij}^k | 1 \leq j, k \leq r) F$ where F is a vector valued function of size r is called a Pfaffian system and $\{u_i\}$ is called a basis of the Pfaffian system.

Bases can be described by those of simpler quotients, of which denominator ideals are nothing but Gröbner deformations of $H_A[s]$, as in the following theorem.

Theorem 1 *Let $w \in \mathbf{Z}^n$ be a generic weight vector for the affine toric ideal I_A such that $\deg_{\mathbf{w}}(I_A) = \deg I_A$. Let u_1, \dots, u_r be a monomial basis of $R_n/(R_n J)$ where the left ideal J is generated by $\text{in}_{\mathbf{w}}(I_A)$ and $E_i - s_i$, $i = 1, \dots, d$ in R_n . Then, the set $\{u_1, \dots, u_r\}$ is a basis of the vector space $R_n/(R_n H_A[s])$.*

Proof. We denote by r the normalized volume of A . Since s_i are indeterminates, the holonomic rank of J and $H_A[s]$ are r by Adolphson's theorem (see, e.g., [1], [15]). In other words, we have $\dim_{\mathbf{C}(s, x)} R_n/(R_n H_A[s]) = r$ and $\dim_{\mathbf{C}(s, x)} R_n/J = r$.

We may assume that u_i are expressed as a monomial in terms of Euler operators $\theta_j = x_j \partial_j$. When we regard J as a system of linear differential equations, it has r linearly independent solutions of the form x^ρ where $\rho \in \mathbf{C}(s)^n$. We denote them by $g_i = x^{\rho(i)}$, $i = 1, \dots, r$. Since g_i are linearly independent solutions, the Wronskian determinant $\det(u_i \bullet g_j)$ is not identically equal to 0. The solution g_j can be extended to a solution f_j of $H_A[s]$ such

that g_j is the leading monomial of f_j with respect to the weight vector w (see, e.g., [15, Chapters 2 and 3]). The series f_j is expressed as $f_j = g_j \sum_{\ell \in M_j} C_\ell x^\ell$, $C_0 = 1$ where M_j denotes the set of the lattice points in a cone and C_ℓ is a constant belonging to $\mathbf{C}(s)$. The series converge in the space of convergent power series $g_j \cdot \mathcal{O}(U)\{M_j\}$ where U is an open set in the s -space and $\mathcal{O}(U)$ is the space of holomorphic functions on U [13]. We replace x_i by $x_i t^{w_i}$ for all i in f_j and denote by xt^w the vector $(x_1 t^{w_1}, \dots, x_n t^{w_n})$. From the construction algorithm of f_j , we may assume that $f_j(xt^w) = g_j(xt^w)(1 + O(t))$ when $t \rightarrow 0$ as a function of t when x is fixed and s lies in U .

Let us prove $W = \det(u_i \bullet f_j) \neq 0$. We denote by $u_i(\rho(j))$ the constant $(u_i \bullet x^{\rho(j)})/x^{\rho(j)}$. Under this notation, we have $x^{-\rho(j)} u_i \bullet g_j = u_i(\rho(j))$ and

$$(x^{-\rho(j)}(u_i \bullet f_j))(xt^w) = \sum_{\ell \in M_j} u_i(\rho(j) + \ell) C_\ell x^\ell t^{\ell w} \quad (4)$$

Note that $\ell w > 0$ for $\ell \neq 0$ and $\ell \in M_j$. Therefore, we have

$$\det(x^{-\rho(j)} u_i \bullet f_j)(xt^w) = \det(x^{-\rho(j)} u_i \bullet g_j)(xt^w) + O(t) \quad (5)$$

from (4) when x is fixed and $t \rightarrow 0$. This implies that the Wronskian determinant $\det(u_i \bullet f_j) = \left(\prod_j x^{\rho(j)} \right) \det(x^{-\rho(j)} u_i \bullet f_j)$ is not identically equal to 0. Therefore u_i are linearly independent in $R_n/(R_n H_A[s])$. Q.E.D.

Remark 1 *We may expect that when the set $\{u_i\}$ is a set of the standard monomials of J , then it is also a set of the standard monomials for $R_n H_A[s]$. However, it is not true in general. In fact, if the set $\{u_i\}$ is the set of the standard monomials of J for a term order \succ in R_n , then there exist rational functions a_α^j such that $\partial^\alpha - \sum_j a_\alpha^j u_j \in R_n H_A[s]$ for any $\partial^\alpha \notin \{u_i\}$, which can be a Gröbner basis of $R_n H_A[s]$ for a weight vector in the Gröbner cone defined by $\alpha \cdot w > w \cdot (\text{exponent of } u_j)$ for $a_\alpha^j \neq 0$ if the cone is not empty.*

Let M be a monomial ideal in $\mathbf{C}[\partial]$. When M is generated by ∂^α , the distraction $\widetilde{M} \subset \mathbf{C}[\theta]$ is generated by $\prod_{i=1}^n \theta_i(\theta_i - 1) \cdots (\theta_i - \alpha_i + 1)$ where $\theta_i = x_i \partial_i$ [15, p.68]. Put $M = \text{in}_w(I_A)$. Then, the ideal J in the Theorem 1 is generated by \widetilde{M} and $\sum_{j=1}^n a_{ij} \theta_j - s_i$, $i = 1, \dots, d$ [15, Sec. 2.3, Prop. 3.1.5]. This leads us to the following corollary.

Corollary 1 *Retain the assumptions of the Theorem 1. The set of the monomial basis of $\mathbf{C}(s)[\theta]/\widetilde{J}$, where \widetilde{J} is the ideal generated by \widetilde{M} and $\sum a_{ij} \theta_j - s_i$, $i = 1, \dots, d$ in the polynomial ring $\mathbf{C}(s)[\theta]$, gives a basis of $R_n/R_n H_A[s]$ by the replacement $\theta_i = x_i \partial_i$.*

We denote by $(\mathbf{0}, \mathbf{1})$ the weight vector such that x_i has the weight 0 and ∂_i has the weight 1. The initial form $\text{in}_{(\mathbf{0}, \mathbf{1})}(\ell)$ is denoted by $\sigma(\ell)$ following the standard notation of the principal symbol in the analysis. $\sigma(\partial_i)$ is set to ξ_i . For $D(s) = \mathbf{C}(s)\langle x, \partial \rangle$, we have $\sigma(D(s)) = \mathbf{C}(s)[x, \xi]$. The induced map from R_n to $\mathbf{C}(s, x)[\xi]$ is also denoted by σ .

Theorem 2 ([5]) *Suppose that I_A is homogeneous and Cohen-Macaulay (i.e., $\mathbf{C}[\partial]/I_A$ is Cohen-Macaulay). Then, the ideal $\sigma(D(s)H_A[s])$ is generated by*

$$\sigma(I_A) \quad \text{and} \quad \sigma(E_i - s_i) = \sum_{j=1}^n a_{ij} x_j \xi_j, \quad i = 1, \dots, d.$$

We assume that I_A is Cohen-Macaulay in the remaining part of this section. Suppose that the set of monomials p_1, \dots, p_r in ξ is a basis of $\mathbf{C}(s, x)[\xi]/\sigma(D(s)H_A[s])$ as a vector space over $\mathbf{C}(s, x)$. The following proposition gives a different way to give a basis of $R_n/R_n H_A[s]$.

Proposition 1 *The set of operators $\bar{p}_i = p_i|_{\xi \rightarrow \partial}$, $i = 1, \dots, r$ is a basis of $R_n/R_n H_A[s]$ as a vector space over $\mathbf{C}(s, x)$.*

Proof. We note that the dimension of $R_n/R_n H_A[s]$ as the vector space and the dimension of $\mathbf{C}(s, x)[\xi]/\sigma(D(s)H_A[s])$ agree and are r . Suppose that it is not a basis. Then, there exist non-zero polynomials $c_i(s, x) \in \mathbf{C}(s, x)$ such that $0 \neq \sum c_i \bar{p}_i \in R_n H_A[s]$. By using the relation $R_n H_A[s] \cap D(s) = D(s)H_A[s]$ (Theorem 4), we may assume that $c_i \in \mathbf{C}[s, x]$ and $\sum c_i \bar{p}_i \in H_A[s]$. Take the principal symbol of it, then we have

$$\sigma\left(\sum c_i \bar{p}_i\right) = \sum_{i \text{ such that } (\mathbf{0}, \mathbf{1})\text{-order of } p_i \text{ is maximal}} c_i p_i(\xi) =: p.$$

By the Theorem 2, we have $p \in \sigma(D(s)H_A[s])$. It contradicts to that p_i , $i = 1, \dots, r$ is a basis. Q.E.D.

We denote the toric ideal $\langle \xi^u - \xi^v \mid Au = Av \rangle$ in the ξ -space also by I_A as long as no confusion arises. A natural question coming from the discussions above is a relation between bases of $\mathbf{C}(s, x)[\xi]/\mathbf{C}(s, x)\langle I_A, \sum_j a_{ij} x_j \xi_j \mid i = 1, \dots, d \rangle$ and of the simpler quotient $\mathbf{C}(s)[\theta]/\mathbf{C}(s)[\theta](\text{in}_w(I_A), \sum_j a_{ij} \theta_j - s_i \mid i = 1, \dots, d)$.

Giving the weight 1 to ∂_i 's, we define $F_k = \{\sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha \mid a_\alpha \in K(x)\}$ where K is $\mathbf{C}(s)$. Let I be a left ideal in R_n . The map σ gives an isomorphism between K -vector spaces

$$\sigma : F_k / (I \cap F_k + F_{k-1}) \longrightarrow \sigma(F_k) / (\sigma(I) \cap \sigma(F_k) + \sigma(F_{k-1}))$$

for $k = 0, 1, 2, \dots$. Note that the set of the standard monomials of R_n/I with respect to an order \succ_1 gives a basis of $\text{gr}(R_n/I) = \bigoplus_{k=0}^{\infty} F_k / (I \cap F_k + F_{k-1})$. Let γ be a map which translate a basis $\{u_i\}$ in the Theorem 1 to a basis of $\text{gr}(R_n/R_n H_A[s])$. Note that when u_i 's are the standard monomials of $R_n H_A[s]$ with respect to an order \succ_1 , we have $\gamma = \text{id}$. Then, we have a $\mathbf{C}(s, x)$ -linear isomorphism

$$\sigma \circ \gamma : R_n / (R_n H_A[s]) \rightarrow \text{gr}(R_n / R_n H_A[s]) \rightarrow \text{gr}(\sigma(R_n) / \sigma(R_n H_A[s])). \quad (6)$$

By using the map γ , we obtain the following theorem in the commutative algebra.

Theorem 3 *Suppose that the affine toric ideal is homogeneous and Cohen-Macaulay. Let w be a generic weight vector for the toric ideal I_A . Put $M = \text{in}_w(I_A)$.*

1. *Let \widetilde{M} be the distraction of M which lies in the commutative ring $\mathbf{C}(s)[\theta]$. Then, a basis of the quotient space of $\mathbf{C}(s)[\theta]$ by the ideal generated by M and $\sum a_{ij}\theta_j - s_i$, $i = 1, \dots, d$ gives a basis of $\mathbf{C}(x)[\xi]/\langle I_A, \sum a_{ij}x_j\xi_j \mid i = 1, \dots, d \rangle$ as the $\mathbf{C}(s)$ -vector space by taking the image by $\sigma \circ \gamma$.*
2. *If $M = \text{in}_w(I_A)$ is square free, then a basis of the quotient space of $\mathbf{C}[\xi]$ by the ideal generated by M and the parameter system $\sum a_{ij}\xi_j$, $i = 1, \dots, d$ gives a basis of $\mathbf{C}(x)[\xi]/\langle I_A, \sum a_{ij}x_j\xi_j \mid i = 1, \dots, d \rangle$ by taking the image by $\sigma \circ \gamma$.*

Proof. (1) Applying the Corollary 1 and the Theorem 2, we obtain the conclusion by the definition (6). (2) When M is square free, \widetilde{M} agrees with M . Then, (2) follows from (1). Q.E.D.

3 $H_A[s]$ as the Annihilating Ideal of Hypergeometric Series

We denote by $(-w, w, 0)$ the weight vector in $D[s]$ such that the weight $-w_i$ stands for x_i , w_i for ∂_i , and 0 for all s_i 's. Let \prec be a block lexicographic order such that $s_1 \succ s_2 \succ \dots \succ s_d \succ$ other variables. The order for "other variables" may be any term order.

Proposition 2 ([15, Theorem 3.1.3]) *Let G_A be a Gröbner basis for the homogeneous toric ideal I_A for the order $\prec_{(-w, w, 0)}$. Then the set*

$$G_A \cup \{s_i - E_i \mid i = 1, \dots, d\}$$

is a Gröbner basis of $H_A[s]$ in $D[s]$ for the order $\prec_{(-w, w, 0)}$.

A proof of this proposition can be done by the Buchberger algorithm in the homogenized Weyl algebra $D[h, s]$ where the weight for the homogenization of s_i is 2 and the weight for other variables is 1.

Let w be a generic weight. When I_A is homogeneous, there exist $r = \text{vol}(A)$ independent solutions of the form $x^{\rho_i(s)}$, $i = 1, \dots, r$ for the initial system $\text{in}_{(-w, w, 0)}(H_A[s]) = \langle \text{in}_w(I_A), s_i - E_i \mid i = 1, \dots, d \rangle$.

Proposition 3 *Let $\ell \in D[s]$ be a differential operator which annihilates all $x^{\rho_i(s)}$. Then, $\ell \in \text{in}_{(-w, w, 0)}(H_A[s])$.*

Proof. Using $E_j - s_j$, we can delete the variable s_j from ℓ . Then, we may assume that the operator ℓ does not contain the variable s . We decompose the operator ℓ into a sum of Euler operators as $\ell = \sum_{\alpha} x^{\alpha} \ell_{\alpha}(\theta)$ where $\theta_j =$

$x_j \partial_j$. Since we have $\ell \bullet x^{\rho(s)} = \sum \ell_\alpha(\rho(s)) x^{\alpha+\rho(s)} = 0$, we may assume that $\ell_\alpha(\theta) \bullet x^{\rho_i(s)} = 0$ for any i and α . This leads a system of algebraic equations $\ell_\alpha(\rho_i(s)) = 0$ in $\mathbf{C}[s]$. We can show that $\ell_\alpha(\theta) \in D \cdot \text{in}_w(I_A)$. Since $D \cdot \text{in}_w(I_A) \cap \mathbf{C}[\theta](\text{the distraction}) + (A\theta - s)$ is a radical ideal of which number of solutions is r [15, Lemma 3.2.4, Theorem 3.2.10], we obtain the conclusion. Q.E.D.

The solution $x^{\rho_i(s)}$ of the initial system can be extended to a solution of $H_A[s]$. We denote the solution by f_i . The following theorem is shown by L. Matusevich [11]. We give a slightly different proof.

Theorem 4 ([11]) *Let L be a differential operator in $D[s]$. If $L \bullet f_i = 0$ for any $i \in \{1, 2, \dots, r\}$, then $L \in H_A[s]$.*

Proof. We express the operator L as $L = \sum L_\alpha s^\alpha$, $L_\alpha \in D$. Using $E_j - s_j$, we can delete s_j from L . Therefore, we may assume $L \in D$. The operator L can be expressed as $\sum x^p \partial^q$. Let G_A be a Gröbner basis of I_A for the generic weight vector w . By reducing each ∂^q by the Gröbner basis G_A , we may assume that

$$L = \sum_{\partial^q \text{ is a standard monomial}} x^p \partial^q.$$

Suppose that $L \neq 0$. Since $L \bullet f_i = 0$, we have

$$\text{in}_{(-w, w, 0)}(L) \bullet \text{Start}_w(f_i) = 0.$$

We note that $\text{Start}_w(f_i) = x^{\rho_i(s)}$. We apply the Proposition 3. Then, we have $\text{in}_{(-w, w, 0)}(L) \in \text{in}_{(-w, w, 0)}(H_A[s])$. Let \succ be a term order such that $s_1 \succ \dots \succ s_d \succ (\text{other variables})$. The monomial operators $\text{in}_w(G_A)$ and the Euler operators $\overline{s_j} - E_j$ are a Gröbner basis for the order $\succ_{(-w, w, 0)}$ (Prop. 2). Since, $\text{in}_{(-w, w, 0)}(L)$ is irreducible by this Gröbner basis, we conclude that $L = 0$. Q.E.D.

4 A -hypergeometric Systems for Order Polytopes

First, recall what the order polytope of a finite partially ordered set ([9, p. 115]) is. Let $P = \{a_1, \dots, a_d\}$ be a finite partially ordered set with $|P| = n$. A *poset ideal* of P is a subset α of P such that if $a \in \alpha$ and $b \in P$ together with $b \leq a$, then $b \in \alpha$. Thus in particular the empty set as well as P itself is a poset ideal of P . Let $\mathcal{J}(P)$ denote the distributive lattice ([9, p. 118]) consisting of all poset ideals of P , ordered by inclusion. For Example, if P is the disjoint union of two chains of length 2 and length 3 drawn in Figure 1, then $L = \mathcal{J}(P)$ is the distributive lattice drawn in Figure 2.

Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ denote the standard unit coordinate vectors of \mathbf{R}^d . If β is a subset of P , then we write w_β for the $(0, 1)$ -vector $\sum_{a_i \in \beta} \mathbf{e}_i \in \mathbf{R}^d$. The *order polytope* $\mathcal{O}(P) \subset \mathbf{R}^d$ of P is the convex hull of the finite set $\{w_\alpha : \alpha \in \mathcal{J}(P)\}$. Its dimension is $\dim \mathcal{O}(P) = d$.

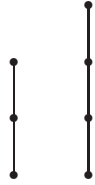


Figure 1:

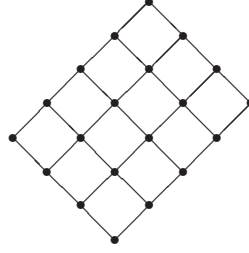


Figure 2:

Let $K = \mathbf{C}(\{\xi_\alpha\}_{\alpha \in \mathcal{J}(P)})$ denote the rational function field in $|\mathcal{J}(P)|$ variables over \mathbf{C} . Let $A = K[t_1, \dots, t_d, s]$ denote the polynomial ring in n variables over K . If β is a subset of P , then we write u_β for the squarefree monomial $\prod_{a_i \in \beta} t_i s$. Let $K[\mathcal{O}(P)]$ denote the subalgebra of A which is generated by those squarefree monomials u_β with $\beta \in \mathcal{J}(P)$. The semigroup ring $K[\mathcal{O}(P)]$ is introduced by [8]. We call $K[\mathcal{O}(P)]$ the *toric ring* of $\mathcal{O}(P)$. The Krull-dimension of $K[\mathcal{O}(P)]$ is $d + 1$.

Let $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ denote the polynomial ring in $|\mathcal{J}(P)|$ variables over K and define the surjective ring homomorphism $\pi : K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}] \rightarrow K[\mathcal{O}(P)]$ by setting $\pi(x_\alpha) = u_\alpha$. Its kernel $I_{\mathcal{O}(P)}$ is called the *toric ideal* of $\mathcal{O}(P)$. It is known [8] that $I_{\mathcal{O}(P)}$ is generated by those quadratic binomials

$$x_\alpha x_\beta - x_{\alpha \wedge \beta} x_{\alpha \vee \beta} \quad (7)$$

such that α and β are incomparable in the distributive lattice $\mathcal{J}(P)$. We fix an ordering $<$ of variables of $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ with the property that if $\alpha > \beta$ in $\mathcal{J}(P)$, then $x_\alpha < x_\beta$. Let $<_{\text{rev}}$ be the reverse lexicographic order on $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]$ induced by the ordering $<$. In [8] it is shown that the set of binomials (7) is the reduced Gröbner basis of $I_{\mathcal{O}(P)}$ with respect to $<_{\text{rev}}$. Thus $\text{in}_{<_{\text{rev}}}(I_{\mathcal{O}(P)})$ is generated by those squarefree quadratic monomial $x_\alpha x_\beta$ such that α and β are incomparable in $\mathcal{J}(P)$.

Let

$$\theta_i = \sum_{\alpha_i \in \alpha} \xi_\alpha x_\alpha, \quad 1 \leq i \leq d$$

together with

$$\theta_0 = \sum_{\alpha \in \mathcal{J}(P)} \xi_\alpha x_\alpha.$$

It follows that the sequence $(\theta_0, \theta_1, \dots, \theta_d)$ is a system of parameters of both the residue rings $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]/I_{\mathcal{O}(P)}$ and $K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]/\text{in}_{<_{\text{rev}}}(I_{\mathcal{O}(P)})$. A fundamental question is to find the standard monomials of the 0-dimensional residue ring

$$K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]/(\text{in}_{<_{\text{rev}}}(I_{\mathcal{O}(P)}), \theta_0, \theta_1, \dots, \theta_d). \quad (8)$$

In general, however, the question is very hard. When P is the disjoint union of two chains, a complete answer can be given.

Let $P_{p,q}$ denote the disjoint union of two chains $C_p : a_1 < \dots < a_p$ of length $p-1$ and $C_q : b_1 < \dots < b_q$ of length $q-1$. Let $\alpha_{i,j}$, where $0 \leq i \leq p$ and $0 \leq j \leq q$, be the poset ideal $\{a_1, \dots, a_i, b_1, \dots, b_j\}$. In particular $\alpha_{0,0} = \emptyset$. Write $\xi_{i,j}$ for $\xi_{\alpha_{i,j}}$ and $x_{i,j}$ for $x_{\alpha_{i,j}}$. Let

$$\theta_{i*} = \sum_{i \leq k \leq p, 0 \leq j \leq q} \xi_{k,j} x_{k,j}, \quad 0 \leq i \leq p$$

and

$$\theta_{*j} = \sum_{0 \leq i \leq p, j \leq \ell \leq q} \xi_{i,\ell} x_{i,\ell}, \quad 0 \leq j \leq q.$$

Thus in particular

$$\theta_{0*} = \theta_{*0} = \sum_{0 \leq i \leq p, 0 \leq j \leq q} \xi_{i,j} x_{i,j}.$$

Let $K[\mathbf{x}] = K[\{x_{i,j}\}_{0 \leq i \leq p, 0 \leq j \leq q}]$. Then the residue ring (8) of $P_{p,q}$ is $K[\mathbf{x}]/J_{p,q}$, where

$$J_{p,q} = (\{x_{i,j} x_{k,\ell}\}_{i < k, \ell < j}, \{\theta_{i*}\}_{0 \leq i \leq p}, \{\theta_{*j}\}_{0 \leq j \leq q}).$$

Let $<_{\text{rev}}$ denote the reverse lexicographic order on $K[\mathbf{x}]$ induced by the ordering of the variables

$$x_{0,0} > x_{1,0} > x_{0,1} > x_{2,0} > x_{1,1} > x_{0,2} > \dots > x_{p,q}.$$

Lemma 1 *In $K[\mathbf{x}]/\text{in}_{<_{\text{rev}}}(J_{p,q})$ one has $x_{i,j} x_{i',j'} = x_{i,j} x_{i',j} = 0$ together with $x_{i,0} = x_{0,j} = 0$.*

Proof. Let $i < i'$. Then

$$\theta_{i'*} x_{i,j} - x_{i,j} \left(\sum_{i' \leq k, j \leq \ell} \xi_{k,\ell} x_{k,\ell} \right)$$

belongs to $(\{x_{i,j} x_{k,\ell}\}_{i < k, \ell < j})$. Hence

$$x_{i,j} \left(\sum_{i' \leq k, j \leq \ell} \xi_{k,\ell} x_{k,\ell} \right)$$

belongs to J . Thus its initial monomial $x_{i,j} x_{i',j}$ belongs to $\text{in}_{<_{\text{rev}}}(J_{p,q})$. Let $i = i'$. The initial monomial of

$$\theta_{i*} x_{i,i} - \xi_{i,i}^{-1} \theta_{*i} \sum_{k=0}^{i-1} \xi_{i,k} x_{i,k}$$

is $x_{i,i}^2$. Hence $x_{i,i}^2 \in \text{in}_{<_{\text{rev}}}(J_{p,q})$. Similarly, $x_{i,j} x_{i,j'} \in \text{in}_{<_{\text{rev}}}(J_{p,q})$.

Moreover, since θ_{i*} and θ_{*j} belong to J , their initial monomials $x_{i,0}$ and $x_{0,j}$ belong to $\text{in}_{<_{\text{rev}}}(J_{p,q})$. Q.E.D.

Let $\mathcal{S}_{p,q}$ denote the set of squarefree monomials of $K[\mathbf{x}]$ of the form

$$x_{i_1, j_1} x_{i_2, j_2} \cdots x_{i_r, j_r}, \quad (9)$$

where

$$0 < i_1 < i_2 < \cdots < i_r \leq p, \quad 0 < j_1 < j_2 < \cdots < j_r \leq q, \quad r = 0, 1, 2, \dots$$

Theorem 5 *The set of standard monomials of $\text{in}_{<\text{rev}}(J_{p,q})$ is equal to $\mathcal{S}_{p,q}$.*

Proof. Lemma 1 guarantees that each standard monomial must belong to $\mathcal{S}_{p,q}$. In [3] it is proved that the number of standard monomials of degree r coincides with the number of maximal chains of $\mathcal{J}(P)$ with r descents. Recall that the descents of a maximal chain

$$\alpha_{0,0} = \alpha_{i_0, j_0} < \alpha_{i_1, j_1} < \cdots < \alpha_{i_{p+q}, j_{p+q}} = \alpha_{p,q}$$

of $\mathcal{J}(P)$ are those α_{i_k, j_k} with $1 \leq k < p+q$ such that

$$i_{k-1} = i_k < i_{k+1}, \quad j_{k-1} < j_k = j_{k+1}.$$

Now, given a squarefree monomial (9) of degree r , we can associate a unique maximal chain whose descents are

$$\alpha_{i_1-1, j_1}, \alpha_{i_2-1, j_2}, \dots, \alpha_{i_r-1, j_r}.$$

in the obvious way. (See Figure 3.) Hence the number of squarefree monomials (9) of degree r is less than or equal to that of standard monomials of degree r . It then follows that $\mathcal{S}_{p,q}$ is the set of standard monomials of $\text{in}_{<\text{rev}}(J_{p,q})$, as desired. Q.E.D.

Remark 2 *Let s_0, s_{i*}, s_{*j} , where $1 \leq i \leq p$ and $1 \leq j \leq q$, be indeterminates over $K = \mathbf{C}(\{\xi_\alpha\}_{\alpha \in \mathcal{J}(P)})$ and $K' = K(s_0, \{s_{i*}\}_{1 \leq i \leq p}, \{s_{*j}\}_{1 \leq j \leq q})$. Let $\theta'_{i*} = \theta_{i*} - s_{i*}$ and $\theta'_{*j} = \theta_{*j} - s_{*j}$, where $1 \leq i \leq p$ and $1 \leq j \leq q$, together with $\theta'_{0*} = \theta_{0*} - s_0$ and $\theta'_{*0} = \theta_{*0} - s_0$. Then the proof of Lemma 1 as well as that of Theorem 5 is valid for the ideal $J'_{p,q}$ of $K'[\mathbf{x}]$, where*

$$J'_{p,q} = (\{x_{i,j} x_{k,\ell}\}_{i < k, \ell < j}, \{\theta'_{i*}\}_{0 \leq i \leq p}, \{\theta'_{*j}\}_{0 \leq j \leq q}),$$

and the set of standard monomials of $\text{in}_{<\text{rev}}(J'_{p,q})$ is also equal to $\mathcal{S}_{p,q}$.

Remark 3 *For $J_{p,q}$, $p = 1$, the map γ in (6) can be the identity map. Then, $S_{1,q}$ gives a K -basis of the vector space*

$$K[\{x_\alpha\}_{\alpha \in \mathcal{J}(P)}]/(I_{\mathcal{O}(P)}, \theta_0, \theta_1, \dots, \theta_d)$$

where $P = P_{1,q}$. Let us prove it. We have $\mathcal{S}_{1,q} = \{1, x_{1,1}, x_{1,2}, \dots, x_{1,q}\}$ then, by the Corollary 1, $\{1, \partial_{1,1}, \partial_{1,2}, \dots, \partial_{1,q}\}$ is a basis of the Pfaffian system. For any ∂^α , of which order is larger than 1, there exist $a_j^\alpha \in \mathbf{C}(s, x)$ such that $\underline{\partial^\alpha} -$

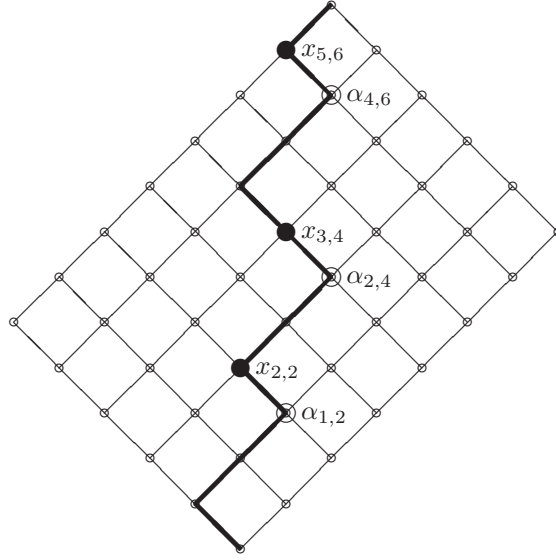


Figure 3:

$a_0^\alpha - \sum_{j=1}^q a_j^\alpha \partial_{1,j} \in R_n H_A[s]$. There are also operators of the forms $\underline{x_{0,j}} \partial_{0,j} + x_{1,j} \partial_{1,j} - (\text{a constant})$, $\underline{x_{0,1}} \partial_{0,1} + \sum_{j=1}^q x_{1,j} \partial_{1,j} - (\text{a constant})$ in $R_n H_A[s]$. They are a Gröbner basis in $\overline{R_n}$ with the (graded) reverse lexicographic order \succ_1 . The leading terms are underlined. By the Remark 1 and the Theorem 3, we obtain the conclusion.

We now turn to the discussion of the normalized volume of order polytopes. It follows from [17] that the normalized volume of the order polytope $\mathcal{O}(P)$ is equal to $e(P)$, the number of linear extensions of P . Recall that an *antichain* of P is a subset B of P such that if a and b belong to B with $a \neq b$, then a and b are incomparable in P . The *width* of P is the supremum of cardinalities of antichains of P . The *length* of a chain C is $|C| - 1$. The *rank* of P is the supremum of lengths of chains of P .

Lemma 2 Fix positive integers d and r . Let P be the disjoint union of d chains C_1, \dots, C_d and suppose that the length of each chain C_i with $1 \leq i < d$ is at most $r - 1$. Then there exists a polynomial $f(n)$ in n of degree $r(d - 1)$ such that $e(P)$ is at most $f(n)$, where $n = |P|$.

Proof. Let ℓ_i denote the length of C_i . Then the number of linear extensions of P is

$$e(P) = \binom{n}{\ell_1, \ell_2, \dots, \ell_d} = \frac{n!}{\ell_1! \ell_2! \dots \ell_d!}.$$

Since $\ell_d = n - \sum_{i=1}^{d-1} \ell_i \geq n - r(d-1)$, it follows that

$$e(P) \leq \frac{n!}{\ell_d!} \leq \frac{n!}{(n - r(d-1))!}.$$

Let

$$f(n) = n(n-1)(n-2) \cdots (n - r(d-1) + 1),$$

which is a polynomial in n of degree $r(d-1)$. Then $e(P) \leq f(n)$, as required. Q.E.D.

Theorem 6 *Fix positive integers d and r . Let P be a finite partially ordered set and suppose that there exists a chain C of P such that*

- (i) *the width of $P \setminus C$ is at most $d-1$;*
- (ii) *the rank of $P \setminus C$ is at most $r-1$.*

Then there exists a polynomial $f(n)$ in n of degree $r(d-1)$ such that $e(P)$ is at most $f(n)$, where $n = |P|$.

Proof. Since the width of $P \setminus C$ is at most $d-1$, Dilworth's theorem [4] guarantees the existence of $d-1$ chains C_1, \dots, C_{d-1} of $P \setminus C$, where the length of each C_i is at most $r-1$, such that $P \setminus C = C_1 \cup C_2 \cup \cdots \cup C_{d-1}$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. Hence there exists a partially ordered set Q which is the disjoint union of d chains C'_1, \dots, C'_d , where the length of each C'_i with $1 \leq i < d$ is at most $r-1$, such that there is an order preserving bijection $\varphi : Q \rightarrow P$. Hence $e(P) \leq e(Q)$. Thus the desired result follows from Lemma 2. Q.E.D.

Let us come back to our discussion on the A -hypergeometric system. We note that we have used a standard convention of variable names in combinatorics and commutative algebra, which is not compatible with that in A -hypergeometric systems and D -modules. Here is a table of the correspondence.

Combinatorics	A -hypergeometric system
x	ξ or ∂
ξ	x
θ stands for a parameter system.	θ stands for the Euler operator

The results of this section together with those in the Section 2 yield the following claims on Pfaffian systems of A -hypergeometric systems.

1. By Theorem 5 and Corollary 1, the set $\mathcal{S}_{p,q}|_{x_{ij} \rightarrow \partial_{ij}}$ gives a basis of the Pfaffian system for the A -hypergeometric system defined by the order polytope $\mathcal{O}(P_{p,q})$, which agrees with the Aomoto-Gel'fand system $E(p+1, p+1+q+1)$ [2].
2. By Theorem 6, we can regard the rank $e(P)$ of the hypergeometric system associated to the order polytope $\mathcal{O}(P)$ ($n = |P|$) has the polynomial growth property with respect to n .

5 Bases of Twisted Cohomology Groups

Let $A_1 = (a_1, \dots, a_{n_1}), \dots, A_k = (a_{n_{k-1}+1}, \dots, a_{n_k})$, $a_i \in \mathbf{Z}^m$. To each matrix A_j we associate a generic polynomial $f_j(x, t) = \sum_{i=n_{j-1}+1}^{n_j} x_i t^{a_i}$. For parameters $\alpha_1, \dots, \alpha_k$ and $\gamma_1, \dots, \gamma_m$, we consider the integral

$$\Phi(\alpha, \gamma; x) = \int_C P(x, t) dt_1 \cdots dt_m, \quad P(x, t) = \prod_{j=1}^k f_j(x, t)^{\alpha_j} t^{\gamma} \quad (10)$$

for a suitable twisted cycle C . The function Φ is satisfied by the A -hypergeometric system for

$$A = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ a_1 & \cdots & a_{n_1} & a_{n_1+1} & \cdots & a_{n_2} & & a_{n_{k-1}+1} & \cdots & a_{n_k} \end{pmatrix}$$

and $\beta = (\alpha_1, \dots, \alpha_k, -\gamma_1 - 1, \dots, -\gamma_m - 1)^T$ [6]. We regard P as a function in $t = (t_1, \dots, t_m)$ with the parameter vector x . Put $D(P) = P|_{\alpha=\gamma=1}$. Define the connection ∇ with rational function coefficients by

$$\nabla = d + \sum_{j=1}^m \left(\frac{\partial P}{\partial t_j} / P \right) dt_j \quad (11)$$

where d is the exterior derivative with respect to the variables t_1, \dots, t_m . When $f_j = \sum x_i t^{a_i}$'s are linear polynomials in a generic position, it is well-known that the space of the cohomology group $H^m(\Omega^\bullet(*D(P)), \nabla)$ is isomorphic to the germ of the solution space of the A -hypergeometric system when the parameter x lies in a domain out of the singular locus and $\alpha_i, \gamma_i \in \mathbf{C} \setminus \mathbf{Z}$. This isomorphism gives the following Theorem from our Theorems 1 and 5,

Theorem 7 *Let A be the matrix standing for the order polytope $\mathcal{O}(P_{p,q})$ and (u_1, \dots, u_r) be the set of standard monomials given in Theorem 5. Then, the set of the rational forms*

$$\frac{u_i \bullet P}{P} dt_1 \cdots dt_m, \quad i = 1, \dots, r \quad (12)$$

*is a basis of the twisted cohomology group $H^m(\Omega^\bullet(*D(P)), \nabla)$ when $\alpha_i, \gamma_j \in \mathbf{C} \setminus \mathbf{Z}$.*

This theorem is a different presentation of the celebrated work of K. Aomoto who gave a basis of the twisted cohomology group associated to a generic hyperplane arrangement (see, e.g., [2, Theorem 9.6.2]). 2012/

6 Algorithms

Let I be a zero dimensional left ideal in the ring of differential operators with rational function coefficients, which we denote by $R_n = K(y_1, \dots, y_n)\langle \partial_1, \dots, \partial_n \rangle$, or $R_n(K)$, where K is a field of characteristic 0 and $\partial_i = \partial/\partial y_i$. We also denote the ring of differential operators in the x -variable by R_n as in the previous sections as long as no confusion arises. We assume that the holonomic rank of I is r . We denote by $F = (u_1, \dots, u_r)^T$ a column vector of which elements form a basis of the vector space R_n/J over the field $K(y) = K(y_1, \dots, y_n)$ (F is not a vector valued function in this section). A Pfaffian system of factored form [10] for I and F is $Q_i \partial_i F - P_i$, $i = 1, \dots, n$ satisfying

$$Q_i \partial_i F \equiv P_i F \pmod{J} \quad (13)$$

where Q_i and P_i are $r \times r$ matrix with polynomial entries and Q_i is invertible. The operator ∂_i in the definition may be replaced by $\theta_i = y_i \partial_i$. Let G be a Gröbner basis of I and F a vector of which elements form a set of the standard monomials of G . Then, the matrix $Q_i^{-1} P_i$ can be obtained by computing the normal form of $\partial_i F$ by G . This matrix is called the companion matrix in the theory of Gröbner basis. However, the computation of Gröbner basis in R_n is very heavy task and we can derive a Pfaffian system only for relatively small sized I .

Let us consider the problem of transforming an A -hypergeometric system into a Pfaffian system. The translation is possible by computing Gröbner basis of $R_n H_A[s]$ as we have explained. We propose two efficient methods which work for A -hypergeometric systems.

The first method is to utilize the rank theorem and the Hilbert driven algorithm [18] in the simplest form.

Algorithm 1

Input: a $d \times n$ matrix A with integer entries such that $\mathbf{Z}A = \sum_{j=1}^n \mathbf{Z}(a_{ij} \mid i = 1, \dots, d)$ is a free \mathbf{Z} -module of rank d .

Output: a Pfaffian system.

1. Evaluate the rank r of $H_A[s]$ by the normalized volume of A .

$r = \text{the normalized volume of } A.$

2. Apply the Buchberger algorithm for $H_A[s]$ in R_n . We stop the computation when the Hilbert function is equal to r . Let the output be G .
3. Construct the Pfaffian system $\partial_i - Q_i^{-1} P_i$ by the normal form algorithm in R_n by G .

The set G is a Gröbner basis since the holonomic rank of $R_n H_A[s]$ is r by the Adolphson's theorem [1]. Therefore this algorithm outputs the correct answer.

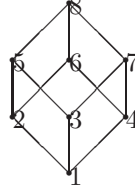


Figure 4: C_1

Example 1 When the point configuration is defined by the lattice in the Figure 4, the matrix A (the vertices of the order polytope associated to the figure) is

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Let us derive a Gröbner basis of $R_n H_A[s]$ and Pfaffian systems by the Algorithm 1. The Hilbert driven Buchberger algorithm takes 8.22s and it takes 168.86s to obtain a Pfaffian system (for all directions $\partial_1, \dots, \partial_8$) from the Gröbner basis. If we do not use the Hilbert driven method, the Buchberger algorithm takes 3457.56s. It is 421 times slower than our algorithm. (The timing data are taken on a machine with the AMD64 CPU (3.33GHz) and 32GB memory, the computer algebra system Risa/Asir and the package yang.rr [14].)

The second method is an algorithm of deriving a Pfaffian system *without* computing a Gröbner basis of the A -hypergeometric ideal $J = R_n H_A[s]$ where $K = \mathbf{C}(s)$. We use series solutions of the A -hypergeometric system to construct a Pfaffian system. The correctness of our method follows from the discussions in the sections 2 and 3. We list facts on $H_A[s]$ which are used in our algorithm.

1. A $(-w, w, 0)$ -Gröbner basis of the hypergeometric ideal $H_A[s]$ in $D[s]$ consists of the Euler operators $E_i - s_i$ and a w -Gröbner basis of the toric ideal I_A [15, Theorem 3.1.3]. Here, $-w$ is the weight for x and w for ∂ , and 0 for the variable s (Prop. 2).
2. If we have a Gröbner basis of I_A , then we can efficiently construct series solutions of $H_A[s]$ (see, e.g., [15, Theorem 2.5.14, Section 3.4] or [7, Theorem 6.12.14]).

Let f be a series solution of $H_A[s]$. We may assume that f belongs in $x^{\rho'} K[[C^* \cap \mathbf{Z}^n]]$ where $K = \mathbf{C}(s)$, $\rho' \in K$ and C is a Gröbner cone defined by a Gröbner basis of I_A [15, Theorem 2.5.14]. We construct a unimodular cone U

contained in the Gröbner cone C by the algorithmic procedure for the resolution of singularities of toric varieties (see, e.g., [16]). The dual U^* is also a unimodular cone. Let $m(i)$ be rays of U^* such that $U^* \cap \mathbf{Z}^n = \sum \mathbf{Z}_{\geq 0} m(i)$. The matrix $(m(1), \dots, m(n))$ is a unimodular matrix. By a unimodular transformation of monomials $y_i = x^{m(i)}$, we may assume that f in terms of y belongs to $y^\rho K[[y_1, \dots, y_n]]$.

Example 2 Using the resolution algorithm should be the final strategy after all other methods fail. Let us find a unimodular cone in a Gröbner cone for the A in the Example 1 without the resolution algorithm. The Gröbner basis for the reverse lexicographic order is

$$\langle x_2x_3 - x_1x_5, x_2x_4 - x_1x_6, x_2x_7 - x_1x_8, x_3x_4 - x_1x_7, x_3x_6 - x_1x_8, \\ x_4x_5 - x_1x_8, x_5x_6 - x_2x_8, x_5x_7 - x_3x_8, x_6x_7 - x_4x_8 \rangle$$

and the Gröbner cone is

$$C = \left\{ w \in \mathbf{R}^8 \left| \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 \end{pmatrix} w \geq 0 \right. \right\}$$

$$= \text{Cone} \left(\begin{pmatrix} -2 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) + \mathbf{R} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{R} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \mathbf{R} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \mathbf{R} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We define c_i ($i = 1, \dots, 9$) by each vector appearing above expression from left to right. In this case, we do not need to apply the resolution algorithm and have a simpler method. In fact, the cone $(\bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4, \bar{c}_5)$ gives an affine toric ideal $\langle a_0a_1a_2 - a_3a_4a_5 \rangle$ where $\bar{c}_0 = (1, \mathbf{0})^T$, $\bar{c}_i = \begin{pmatrix} 1 \\ c_i \end{pmatrix}$, and a_i are variables corresponding to \bar{c}_i . It has the unimodular decomposition because it has a square-free initial term. Matrices $(c_i)_{1 \leq i \leq 9} \setminus c_j$ ($j = 1, 2, 3, 4, 5$) are unimodular and the cone C contains cones generated by them. In the case of $j = 1$, we can take

$$U = \text{Cone}((c_i)_{1 \leq i \leq 9} \setminus c_1) = \text{Cone} \begin{pmatrix} -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its dual cone U^* is

$$U^* = \text{Cone} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \\ -2 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}.$$

Let Y^k be an ideal in $\mathbf{C}[y]$ generated by the monomials of degree k . In the sequel, we assume that a Pfaffian system is expressed in terms of the variable y . We assume the elements of F are monomials of Euler operators. We denote by $F(f)$ a vector obtained by acting operators in F to the series f . θ'_i is the operator in the y variable standing for $x_i \partial / \partial x_i$. Then, we have

$$Q_i(\theta'_i F(f)) = P_i F(f). \quad (14)$$

Since $f \in y^\rho \mathbf{C}[[y]]$, any element of $\theta'_i F(f)$ and $F(f)$ belongs to $y^\rho K[[y]]$. Therefore, we have

$$Q_i(y^{-\rho} \theta'_i F(f) \bmod Y^k) \equiv P_i(y^{-\rho} F(f) \bmod Y^k) \quad (15)$$

in the polynomial ring $K[y]$ modulo Y^k for any natural number k . Moreover, multiplying a suitable polynomial $c(s)$ in s to $F(f)$, we may assume that it holds in $\mathbf{C}[s, y]$. In summary, P_i and Q_i appear as a syzygy among truncated solutions as in (15). This leads us the following algorithm to obtain Q_i and P_i .

Algorithm 2

1. Fix a number k_0 . Determine a vector F of monomials of Euler operators, which is a basis, by the Corollary 1 (or other methods in sections 2 and 4).
2. Solve the polynomial syzygy equations

$$\begin{aligned} & q_i \cdot c(s)(y^{-\rho} \theta'_i F(f) \bmod Y^k) - p_i \cdot c(s)(y^{-\rho} F(f) \bmod Y^k) \\ & + \sum_{|\alpha|=k} r_{k,\alpha} \cdot c(s) y^\alpha = 0 \end{aligned} \quad (16)$$

in $\mathbf{C}[s, y]$ for unknowns $q_i, p_i, r_{k,\alpha}$ where q_i, p_i are polynomial row vectors of which length are $r = \text{rank}(H_A[s])$ and f runs in the space of solutions of $H_A[s]$ and k runs from 1 to k_0 . Here, $F(f)$ is expressed in the variable y . Note that $c(s)$ and ρ depends on f and k and are chosen so that $c(s)(y^{-\rho} \theta'_i F(f) \bmod Y^k)$ and $c(s)(y^{-\rho} F(f) \bmod Y^k)$ belong to $\mathbf{C}[s, y]$.

3. Construct matrices Q_i and P_i from the syzygies. If we cannot construct them or Q_i is not invertible, then increase k_0 and go to 2.
4. Apply the candidate $Q_i\theta'_i - P_i$ to the series solution $F(f) \bmod Y^{N+1}$ truncated at the degree $k = N + 1 > k_0$. Determine remaining undetermined constants by setting $(Q_i\theta'_i - P_i) \bullet (F(f) \bmod Y^k) \equiv 0 \bmod Y^k$. If there is no solution, then increase k_0 and go to 2.
5. (Check the candidate.) Reduce $Q_i\theta'_i F - P_i F$ expressed in the variable x by a $(-w, w, 0)$ -Gröbner basis of $H_A[s]$ by the normal form algorithm. The details of this step will be discussed in the algorithm 3. If the result is yes, then we are done else, increase k_0 and go to 2.

Remark 4 The operator θ'_i in the Algorithm may be replaced by any operator of the form $\sum y^\alpha p_\alpha(\theta_y)$ where p_α is a polynomial in $\theta_{y_1}, \dots, \theta_{y_n}$ to find a “connection” to the direction “ $\sum y^\alpha p_\alpha(\theta_y)$ ”. This means that the Algorithm 2 can also be used to express the gradient $\text{grad } f_i$ and the Hessian $\text{Hess } f_i$ in terms of $F(f_i)$.

We note that we may solve the syzygy equation by the Gröbner basis in the ring of polynomials or the method of undetermined coefficients. We can discard the syzygy of which order is more than Y^k . It seems to be more efficient to determine the coefficients of q_i and p_i step by step from the low degree. See the Example 4.

The membership decision for the order $\prec_{(-w, w, 0)}$ needs a tangent cone algorithm in general, but that for $H_A[s]$ can be performed without the tangent cone algorithm. In fact, the membership decision in the step 5 in the Algorithm 2 can be performed as follows.

- Algorithm 3**
1. Reduce $\ell \in D[s]$ by $\{s_i - E_i \mid i = 1, \dots, d\}$. In other words, when $\ell = \sum \ell_\alpha s^\alpha$, $\ell_\alpha \in D$, rewrite them to $\ell' := \sum \ell_\alpha \prod E_i^{\alpha_i} \in D$.
 2. Reduce ℓ' by G_A . Set the result to ℓ'' .
 3. If $\ell'' = 0$, then $\ell \in H_A[s]$ (yes) else $\ell \notin H_A[s]$ (no).

We note that these two steps can be performed in finite steps. If the operator ℓ'' is zero, then $\ell \in H_A[s]$ else $\ell \notin H_A[s]$. In fact, when $\ell'' \neq 0$, then $\text{in}_{\prec_{(-w, w, 0)}}(\ell'') \notin D[s] \cdot \{s_1, \dots, s_d, \text{in}_w(G_A)\}$. It follows from Proposition 2 that $\ell'' \notin H_A[s]$.

From Theorems 1 and 4, we obtain the following theorem.

Theorem 8 *The algorithm 2 terminates in finite steps and outputs a correct answer.*

Example 3 In the case of $A = (1, 2)$ and $w = (1, 0)$, the hypergeometric ideal $H_A[s] = \langle \partial_1^2 - \partial_2, \theta_1 + 2\theta_2 - s \rangle$ has two linearly independent series solutions:

$$f_1 = x_2^{\frac{s}{2}} \sum_{k=0}^{\infty} a_{1k} \left(\frac{x_1^2}{x_2} \right)^k, \quad a_{1k} = \frac{\Gamma(\frac{s}{2} + 1)}{\Gamma(2k + 1)\Gamma(\frac{s}{2} - k + 1)}$$

$$f_2 = x_1 x_2^{\frac{s-1}{2}} \sum_{k=0}^{\infty} a_{2k} \left(\frac{x_1^2}{x_2} \right)^k, \quad a_{2k} = \frac{\Gamma(\frac{s-1}{2} + 1)}{\Gamma(2k+2)\Gamma(\frac{s-1}{2} - k + 1)}.$$

The set of standard monomials determined by Theorem 1 is $(1, x_1 \partial_1)$. We make the change of variables $y_1 = \frac{x_1^2}{x_2}$ and $y_2 = x_1$ standing for the unimodular matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, which is the set of the rays of the unimodular cone generated by $(1, 2)$ and $(0, -1)$. The operators $x_1 \partial_1$ and $x_2 \partial_2$ are transformed to $2y_1 \partial_{y_1} + y_2 \partial_{y_2}$ and $-y_1 \partial_{y_1}$ respectively by the transformation. We denote θ_{y_1} by θ_1 and θ_{y_2} by θ_2 . The hypergeometric ideal and its solutions are changed as follows:

$$R_2 H_A[s] = \langle (\theta_2 + 2\theta_1)(\theta_2 + 2\theta_1 - 1) + y_1 \theta_1, \theta_2 - s \rangle,$$

$$f_1 = \left(\frac{y_2^2}{y_1} \right)^{\frac{s}{2}} \sum_{k=0}^{\infty} a_{1k} y_1^k, \quad f_2 = y_2 \left(\frac{y_2^2}{y_1} \right)^{\frac{s-1}{2}} \sum_{k=0}^{\infty} a_{2k} y_1^k.$$

Let us derive the Pfaffian system for a direction θ_1 by following the Algorithm 2 and the Remark 4. We put $k_0 = 1$ and take a vector $F = (1, \theta_1)$ as a basis. We note that $(1, 2\theta_1 + \theta_2)$ is a basis from the theorem and then $(1, \theta_1)$ is a basis. Solving the following syzygy equations:

$$(q_{11} \quad q_{12}) \begin{pmatrix} -s/2 \\ s^2/4 \end{pmatrix} - (p_{11} \quad p_{12}) \begin{pmatrix} 1 \\ -s/2 \end{pmatrix} + r_{1,(1),1} y_1 = 0,$$

$$(q_{11} \quad q_{12}) \begin{pmatrix} -(s-1)/2 \\ (s-1)^2/4 \end{pmatrix} - (p_{11} \quad p_{12}) \begin{pmatrix} 1 \\ -(s-1)/2 \end{pmatrix} + r_{1,(1),2} y_1 = 0,$$

we get the solution

$$\begin{aligned} (q_{11}, q_{12}, p_{11}, p_{12}, r_{1,(1),1}, r_{1,(1),2}) = & \langle (0, -4, s(s-1), 2(s-1), 0, 0), \\ & (1, 0, 0, 1, 0, 0), \\ & (0, 0, 0, 2y_1, s, s-1), \\ & (0, 0, y_1, 0, 1, 1) \rangle. \end{aligned}$$

The 3rd and the 4th syzygies are of degree 1 and they should be discarded. The first syzygy will be extended to a syzygy of degree 1 in the next step. The operator corresponding to the second element is $\theta_1 - \theta_1 = 0 \in H_A[s]$. This yields the trivial relation

$$(1 \quad 0) \theta_1 \begin{pmatrix} 1 \\ \theta_1 \end{pmatrix} \equiv (0 \quad 1) \begin{pmatrix} 1 \\ \theta_1 \end{pmatrix} \pmod{H_A[s]}.$$

The other operators do not belong to $H_A[s]$. Next, we put $k_0 = 2$ and solve above two and the following syzygy equations:

$$(q_{11} \quad q_{12}) \begin{pmatrix} -s(s-2)y_1/8 - s/2 \\ s(s-2)^2 y_1/16 + s^2/4 \end{pmatrix} - (p_{11} \quad p_{12}) \begin{pmatrix} sy_1/4 + 1 \\ -s(s-2)y_1/8 - s/2 \end{pmatrix} + r_{2,(2),1} y_1^2 = 0,$$

$$\begin{aligned}
& (q_{11} \quad q_{12}) \begin{pmatrix} -s(s-3)y_1/24 - (s-1)/2 \\ (s-1)(s-3)^2y_1/48 + (s-1)^2/4 \end{pmatrix} \\
& - (p_{11} \quad p_{12}) \begin{pmatrix} (s-1)y_1/12 + 1 \\ -(s-1)(s-3)y_1/24 - (s-1)/2 \end{pmatrix} + r_{2,(2),2}y_1^2 = 0.
\end{aligned}$$

We get the solution

$$\begin{aligned}
& (q_{11}, q_{12}, p_{11}, p_{12}, r_{1,(1),1}, r_{1,(1),2}, r_{2,(2),1}, r_{2,(2),2}) \\
& = \langle (1, 0, 0, 1, 0, 0, 0, 0), \\
& \quad (0, -96, 24s(s-1), 24(y_1 + 4s - 2), 12s, 12(s-1), 3s(s-2), (s-1)(s-3)), \\
& \quad (0, 48(y_1 - 4s + 2), -12s(s-1)(y_1 - 4s + 2), 48(2s-1)^2, 12s(2s-1), 12s(2s-1), \\
& \quad \quad 3s(2s^2 - 5s + 4), s-1, 2s^2 - 7s + 9), \\
& \quad (0, 0, -12y_1^2, 0, 12y_1, 3(sy_1 + 4), (s-1)y_1 + 12) \rangle.
\end{aligned}$$

The 4th syzygy is of degree 2 and it should be discarded. The operator corresponding to the second element is reduced by $H_A[s]$ as follow:

$$\begin{aligned}
-96\theta_1^2 - 24s(s-1) - 24(y_1 + 4s - 2)\theta_1 &= -24\{4\theta_1^2 + s(s-1) + (y_1 + 4s - 2)\theta_1\} \\
&\rightarrow -24\{4\theta_1^2 + \theta_2(\theta_2 - 1) + (y_1 + 4\theta_2 - 2)\theta_1\} \\
&= -24\{(\theta_2 + 2\theta_1)(\theta_2 + 2\theta_1 - 1) + y_1\theta_1\} \\
&\rightarrow 0.
\end{aligned}$$

This yields the new relation and we obtain a Pfaffian system for the direction y_1 :

$$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \theta_1 \begin{pmatrix} 1 \\ \theta_1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ s(s-1) & y_1 + 4s - 2 \end{pmatrix} \begin{pmatrix} 1 \\ \theta_1 \end{pmatrix} \pmod{H_A[s]}.$$

Example 4 When we use a normal form algorithm by a Gröbner basis to obtain a Pfaffian system, the matrix Q_i is always a diagonal matrix. The degree of polynomials in Q_i by our Algorithm 2 may be smaller than the diagonal Q_i .

Let us illustrate it with an example. Put $A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$ and $s = (s_1, 1/3, s_3, s_1 - 1/2)$. Note that $A = \mathcal{O}(P_{1,2}) = 1\text{-simplex} \times 2\text{-simplex}$. We take a weight vector $w = (0, 1, 1, 1, 1, 0)$ and $y = \left(\frac{x_2x_4}{x_1x_5}, \frac{x_3x_5}{x_2x_6}, x_3, x_4, x_5, x_6\right)$. The basis of series solutions is

$$\begin{aligned}
f_1 &= y^{\rho_1} F_1(1/2 - s_1, -1/3, -s_3, 3/2 - s_1; y_1, y_1 y_2), \\
f_2 &= y^{\rho_2} G_2(-s_1, -s_3, 1/2, -5/6; -y_1, -y_2), \\
f_3 &= y^{\rho_3} F_1(-s_3 - 5/6, -s_1, -1/3, 1/6; y_1 y_2, y_2),
\end{aligned}$$

where ρ_i are linear polynomials in s_i 's, F_1 and G_1 are the Appell hypergeometric function F_1 and the hypergeometric function G_1 in Horn's list respectively. Here,

we apply the Algorithm 2 for the direction $\theta'_i = y_1 \partial_{y_1}$. We will denote y_1 by x and y_2 by y . The degree 0 syzygy returns the pair $q = (1, 0, 0), p = (0, 1, 0)$. Here q and p are rows of Q_i and P_i respectively. The degree 1 syzygy returns the pair

$$q = (0, -x + 1, x - 1), p = (0, -\frac{1}{3}(x + 3s_1), s_1).$$

The degree 2 syzygy returns the pair

$$q = (0, 0, 1 - xy), p = (-\frac{1}{2}s_1(2s_1 - 1), \frac{1}{2}((2s_1 - 2s_3 - 1)xy - 2s_1 + 1), -s_1)$$

where we set the truncation degree N to 5. Note that we get junk syzygies comes from lower order syzygies. The space of the degree m syzygies can be regarded as a vector space over the base field. Syzygies which comes from lower order syzygies can be regarded as a subvector space of the space. These junks can be removed by adding an orthogonality condition for the subspace. We have used this method to find the second syzygies by removing junk syzygies generated by the first one.

In summary, we have $Q_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -x + 1 & x - 1 \\ 0 & 0 & 1 - xy \end{pmatrix}$, of which elements have degree 2 at most. On the other hand, if we derive a Pfaffian system with a diagonalized Q_i , we have $Q_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x - 1)(xy - 1) & 0 \\ 0 & 0 & 1 - xy \end{pmatrix}$, of which elements have the degree 3. The Algorithm 2 of utilizing series solutions is sometimes slower than the Algorithm 1 in our current implementation. However, it often outputs smaller Q_i and P_i as we have seen.

An efficient implementation of the Algorithm 2 for larger problems is not easy. This topic together with applications to statistics will be discussed in a forthcoming paper.

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